The Relation Between Variance and Information Rent in Auctions

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Abstract

This paper examines the conventional wisdom, expressed in McAfee and McMillan's (1987) widely cited survey paper on auctions, that links increased variance of bidder values to increased information rent. We find that although the conventional wisdom does indeed hold in their (1986) model of a linear contract auction, this relationship is an artifact of that particular model and cannot be generalized. Using Samuelson's (1987) model, which is similar but allows for unobservable costs, we show that increased variance does not always imply increased information rent. Finally, we give the appropriate measure of dispersion (different from variance) that provides the link between the bidder value distribution and information rent.

Keywords: Dispersion, Information Rent, Variance, Contract Auctions

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1 Introduction

The Revenue Equivalence Theorem states that the expected price paid in a standard auction equals the expectation of the second-highest bidder value (or second-lowest bidder cost in a procurement) whenever bidder beliefs are independent. Thus, ordering bidder values from highest to lowest, the winning bidder's expected surplus is the expected difference between the highest and second-highest order statistics (or second-lowest and lowest in a procurement). The winning bidder's expected surplus is often referred to as *information rent* because this surplus only accrues when bidders have independent information about their values or costs. Because all of the models we consider in this paper are within the independent private values paradigm, we define information rent as follows.[1]

Definition 1 *In an independent private values (procurement) model where the bidder with the highest value (lowest cost) wins, the* ***information rent*** *earned by the winning bidder is:*

$$IR = \left|E\left(v_{(1)} - v_{(2)}\right)\right|$$

where $v_{(1)}$ and $v_{(2)}$ are the first and second highest (lowest) of the bidders' values (costs).

A substantial amount of auction research has focused on tools that auctioneers can use to capture part of the winning bidder's information rent in an attempt to increase auction revenue; examples include reserve prices and entry fees. Pertinent to this paper is the following conventional wisdom offered by McAfee and McMillan (1987) in their widely cited survey on auctions:

> [A] determinant of the strength of the bidding competition is the variance of the distribution of valuations. The larger is this variance, the larger on average is the difference between the highest and the second highest valuation, and so the larger is the economic rent to the winning bidder.

Our paper shows that the relation between the variance of bidder values and information rent is not always monotonic. There are quite ordinary settings in which a lower variance of the distribution

[1] Although there are formulas for information rent based on complicated envelope theorems and incentive compatibility (see Milgrom, 2004 or Krishna, 2002), our focus on independent private values models allows information rent to be expressed as in Definition 1.

of bidder values may, contrary to the conventional wisdom above, increase the winner's information rent. For instance, consider an example where two bidders draw their values from a power distribution function $F_\gamma(v) = v^\gamma$(parameterized by γ), with variance $Var_\gamma = \frac{\gamma}{(1+\gamma)^2(2+\gamma)}$ and information rent $IR_\gamma = \frac{2\gamma}{(1+\gamma)(1+2\gamma)}$. Both Var_γ and IR_γ are continuous and single-peaked, but obtain their maxima at different values of γ. Specifically, Var_γ is maximized at $\gamma = \left(\sqrt{5}-1\right)/2$ while IR_γ is maximized at $\sqrt{2}/2$. These values are listed in Table 1 to show that decreased variance does not necessarily lead to decreased information rent.

Table 1: IR/VAR example

γ	Var_γ	IR_γ
$\left(\sqrt{5}-1\right)/2 \approx 0.61803$	0.09017	0.34164
$\sqrt{2}/2 \approx 0.70711$	0.08963	0.34315
1	0.08333	0.33333

More generally, the family of power distributions is a member of a larger class of distributions for which lower variance does not imply lower information rent. Membership in this class requires that the distributions cannot be *ordered in dispersion*, defined as follows.

Definition 2 *Let X and Y be random variables distributed by F and G, respectively. Then X is* ***smaller in dispersion*** *than Y ($X \leq_{disp} Y$ or $F \leq_{disp} G$) if*

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta \leq 1$. A family of distributions F_γ parameterized by $\gamma \in \Gamma \subset R$ can be ***ordered in dispersion*** *if $F_{\gamma_1} \leq_{disp} F_{\gamma_2}$ holds for all $\gamma_1 < \gamma_2$.*

Intuitively, $\leq_{disp}$ is a measure of variability that requires the difference between any two quantiles of X to be smaller than the corresponding quantiles of Y. Landsberger and Meilijson (1994) have noted that while this definition of dispersion is well known to statisticians, it has seen little application in economics.[2] Our paper provides one such application where a variety of statistical theorems are

[2] Chateauneuf, Cohen, and Meilijson (2004) show how order in dispersion can be used to measure risk in expected utility models.

collected and applied to the relation between the variance of bidder values and information rent.

In a version of McAfee and McMillan's (1986) principal-agents model, we show that information rent, variance, and the dispersion of the reservation values all increase with the share $s \in [0, 1]$ of costs borne by the principal. We then show that a parsimonious addition of unobservable costs, as in Samuelson (1986, 1987), is enough to upset the ordering in dispersion of the reservation values, thereby allowing counterexamples to the conventional wisdom. Thus, our paper clarifies that dispersion—not variance—is the driving force between information rents and the distribution of bidder values.

2 Linear Contract Auctions

A principal must choose one of n risk neutral agents to do a project. Doing the project costs agent i $(= 1, ..., n)$ his observable costs c_i plus his unobservable costs d_i. For example, observable costs might include materials and equipment rental and unobservable costs might include the opportunity cost of forfeiting other work. We assume that c_i and d_i are drawn from the C^2 distribution functions $F(\cdot)$ and $G(\cdot)$, with corresponding density functions $f(\cdot)$ and $g(\cdot)$ on supports $[c_L, c_H]$ and $[d_L, d_H]$. We also assume that each agent's type pair (c_i, d_i) is drawn independently from the other agents' types.[3] We denote order statistics in the usual way: $c_{(1)}$ and $c_{(2)}$ are the lowest and second lowest of the N agents' observable costs and other order statistics are defined similarly.

The principal holds a *linear contract* auction to select the agent who will do the project. In this auction, the principal announces a sharing rate $s \in [0, 1]$ and then each agent i bids $b_i \in R_+$. The agent with the lowest bid wins the contract, does the project, and is paid by the principal a fixed fee equal to his bid plus the share s of his observable costs: $b_i + sc_i$. We emphasize that at the time of the auction both c_i and d_i are privately known to agent i; upon completion of the job only c_i becomes observable. When $s = 0$, the contract is termed *fixed-price* because the winning agent is paid only his bid. When $s = 1$, the contract is termed *cost-plus* because the agent is paid all of his costs plus his bid. When $s \in (0, 1)$ we have an *incentive contract.*[4]

[3] We allow for the possibility that c_i and d_i are correlated for a given agent i.

[4] This type of auction is often used to procure highway construction. See Bajari, Houghton, and Tadelis (2006), as well as the many references therein, for institutional details on road construction auctions and the uncertainties

The winning agent's profit from the auction is

$$\begin{aligned}\pi_{i|s} &= b_i + sc_i - c_i - d_i \\ &= b_i - v_{i|s} \qquad (1)\end{aligned}$$

where

$$v_{i|s} = (1-s)c_i + d_i \qquad (2)$$

We refer to $v_{i|s}$ as agent i's *reservation value* because it is a lower bound for the fixed payment the agent must receive to do the project and, more importantly, because this value will play the same role as a bidder's private value in a standard—as opposed to a linear contract—auction. Since $v_{i|s}$ is a linear combination of two random variables, it is itself a random variable with some distribution $H_s(\cdot)$ and density $h_s(\cdot)$ on support $[v_{L|s} = (1-s)\,c_L + d_L, v_{H|s} = (1-s)\,c_H + d_H]$ that can be computed using $F(\cdot)$ and $G(\cdot)$. In what follows, it will also be helpful to denote the distribution of costs not covered by the principal, $(1-s)c$, as $F_s(\cdot)$.

Our model is equivalent to the model in Samuelson (1987). However, Samuelson emphasizes the effect that unobservable costs have on adverse selection while we are more interested in their effect on information rents. In one sense, our model simplifies the model in McAfee and McMillan (1986) because we omit risk aversion and moral hazard. Yet, our model also extends McAfee and McMillan's by including an idiosyncratic, unobservable cost component in the bidder types, which, as Samuelson (1986, 1987) note, is a more realistic setting.[5] We chose our version of the principal-agents model for two reasons. First, including the unobservable, non-contractible costs allows us to understand the relationship between the sharing parameter, variance, and information rents in as parsimonious a linear contract auction model as possible. Second, by excluding risk aversion and moral hazard, we are able to avoid the equilibrium analysis in McAfee and McMillan (1986) and use the following version of the Revenue Equivalence Theorem.

inherent therein.

[5] McAfee and McMillan (1986) actually consider a second source of uncertainty in that the agent's effort cannot be observed by the prinicpal. Nevertheless, in their model, the agent's optimal effort level is a deterministic function of the sharing rate, so their only idiosynchratic element is the agent's observable cost.

Theorem 3 *(Revenue Equivalence) Fix a particular sharing rate $s \in [0,1]$. Then for any auction in which the agent with the lowest reservation value wins the auction and such that an agent with value $v_{H|s}$ receives an expected payment of 0, the principal's expected payment is:*

$$\begin{aligned} \tau &= E(v_{(2)|s}) + sE(c|v_{(1)|s}) && (3) \\ &= E\left(v_{(2)|s} - v_{(1)|s}\right) + E(c + d|v_{(1)|s}) && (4) \end{aligned}$$

where $v_{(1)|s}$ and $v_{(2)|s}$ are the two lowest reservation values and $E(c|v_{(1)|s})$ and $E(d|v_{(1)|s})$ are the expected observable and unobservable costs of the agent with the lowest reservation value.

Proof. Consider a sealed-bid second-price auction where the agent who submits the lowest bid wins the auction, does the project, and is paid the second-lowest bid plus the fraction s of his observable cost c. As is usual with second-price auctions, it is a weakly dominant strategy for an agent to bid his value $v_{i|s}$. This means that the bidder with the lowest reservation value $v_{(1)|s}$ wins the auction, gets paid the second-lowest bid $v_{(2)|s}$ and is also reimbursed the fraction s of his observable cost. This explains equation (3). Further, the Revenue Equivalence Theorem (Myerson, 1981) states that any auction that awards the good to the lowest value $v_{(1)|s}$ and such that an agent with highest possible value $v_{H|s}$ has an expected payoff of 0 yields the same expected payment to the principal. Adding and subtracting $E(v_{(1)|s}) = E\left[(1-s)\,c + d|v_{(1)|s}\right]$ to equation (3) results in (4). ■

We emphasize that the value of s is fixed in the above theorem. For a fixed value of s, first- and second-price auctions result in the principal making the same expected payment. For different values of s, the principal's expected payment can differ for two reasons. First, the agent who wins the allocation can change, meaning that for two different values of s there may be two different agents who win the auction. It is well known that revenue equivalence is only required to hold between two auctions if both always result in the same winner (see page 66, Krishna, 2002). Second, when the sharing rate s changes, the timing in the Revenue Equivalence Theorem changes. That is, some of the payment made to the winning bidder is made after the auction, once some of the bidder's private information is no longer private. In contrast, the standard Revenue Equivalence Theorem applies in the interim, when all of the bidder's private information remains private and unverifiable.

2.1 With Observable Costs Only

In this section, we assume that $d_i = 0$ (or more precisely, $d_L = d_H = 0$) so that the only source of uncertainty is the agents' observable costs c_i. This assumption enables us to replicate McAfee and McMillan's "bidding competition effect," though in a more transparent manner since we omit risk aversion and moral hazard. In this special case where $v_{i|s} = (1-s)c_i$, we have $H_s(\cdot) = F_s(\cdot)$.

The first issue is how an increase in the sharing rate s affects the variance of the distribution of the reservation values and the information rents. Let $Var(v_{i|s})$ denote the variance of the distribution $H_s(\cdot)$ and $Var(c_i)$ denote the variance of the distribution $F(\cdot)$. Since $v_{i|s} = (1-s)c_i$, we have $Var(v_{i|s}) = (1-s)^2 Var(c_i)$ and $IR_s = E\left(v_{(2)|s} - v_{(1)|s}\right) = (1-s)E\left(c_{(2)} - c_{(1)}\right)$. Clearly, both $Var(v_{i|s})$ and IR_s are decreasing in the sharing rate s. The reason that variance and information rent associated with different distributions of reservation values move in the same direction in this restricted model is not because higher variance causes higher information rents. (Our example in the introduction has already shown that cannot be true.) Rather, it is because the family of distributions $F_s(\cdot)$ is ordered in dispersion by the sharing rate s, due to the following lemma.

Lemma 4 *Let X be a random variable with a strictly increasing distribution $F(\cdot)$ on its support. For every constant $\lambda \in [0,1]$, define $X_\lambda = \lambda X$. Then $\{X_\lambda\}$ is ordered in dispersion by λ.*

Proof. Select any probabilities α and β such that $\alpha < \beta$. For a given $\lambda \in [0,1]$, let F_λ denote the distribution of X_λ. Since

$$\alpha = prob\left\{X \le F^{-1}(\alpha)\right\} = prob\left\{\lambda X \le \lambda F^{-1}(\alpha)\right\} = prob\left\{\lambda X \le F_\lambda^{-1}(\alpha)\right\}$$

it follows that $F_\lambda^{-1}(\alpha) = \lambda F^{-1}(\alpha)$. Similarly, $F_\lambda^{-1}(\beta) = \lambda F^{-1}(\beta)$. Thus, for $\lambda_1 < \lambda_2$ we have

$$F_{\lambda_1}^{-1}(\beta) - F_{\lambda_1}^{-1}(\alpha) = \lambda_1\left[F^{-1}(\beta) - F^{-1}(\alpha)\right] < \lambda_2\left[F^{-1}(\beta) - F^{-1}(\alpha)\right] = F_{\lambda_2}^{-1}(\beta) - F_{\lambda_2}^{-1}(\alpha)$$

showing that X_λ is ordered in dispersion by λ. ■

This lemma shows that the reservation values $v_{i|s} = (1-s)c_i$ are ordered in dispersion by $1-s$. That is, in the restricted model with only observable costs, the higher is s, the more costs the principal covers, and the less dispersed the distribution of reservation values. The following proposition relates the ordering in dispersion to variance and information rent.

Proposition 5 *Let X and Y be random variables distributed by F and G, respectively. Then if $X \leq_{disp} Y$, we have*

1. *$var(X) \leq var(Y)$, subject to existence, and*

2. *$E[X_{(s)} - X_{(r)}] \leq E[Y_{(s)} - Y_{(r)}]$ for all $1 \leq r < s \leq n$, where $X_{(t)}$ is the t^{th} lowest order statistic of n draws from F and $Y_{(t)}$ is t^{th} lowest order statistic of n draws from G.*

Proof. See page 78 of David and Nagaraja (2003). ■

Recall from Definition 1 that $E[X_{(2)} - X_{(1)}]$ and $E[Y_{(2)} - Y_{(1)}]$ are simply the procurement auction information rents obtained under the distributions F and G. Hence, although $var(X) \leq var(Y)$ does not imply $IR(X) \leq IR(Y)$, Proposition 5 says that rather, they both follow if X is smaller in dispersion than Y. Alternatively, if X and Y cannot be ordered in dispersion, then $var(X) < var(Y)$ and $IR(X) > IR(Y)$ may coexist. Proposition 5 sheds light on our example from the introduction where $X \sim F(X) = X^{\sqrt{2}/2}$ and $Y \sim G(Y) = Y^{(\sqrt{5}-1)/2}$ as the family of power distributions $F_\gamma(v) = v^\gamma$ cannot be ordered in dispersion by γ.[6]

Proposition 5 also sheds light on McAfee and McMillan's (1986) "bidding competition effect." Both the variance of reservation values and information rent necessarily decrease when the sharing rate s increases in their model because bidders have reservation values $v_{i|s} = (1-s)c_i$ that are ordered in dispersion by $1-s$. The next section shows that this ordering in dispersion is a simple artifact of their model and that the parsimonious inclusion of unobservable costs can negate the ordering in dispersion of reservation values based on s. Thus, an increase in s does not guarantee a reduction in both variance and information rent.

2.2 With Observable and Unobservable Costs

We now relax the assumption that $d_L = d_H = 0$ so that an agent's reservation value includes unobservable costs: $v_{i|s} = (1-s)c_i + d_i$. This parsimonious extension of the model presented in the previous subsection is enough to show how the example from the introduction applies to linear

[6] To show that the family of power distribitions $F_\gamma(v) = v^\gamma$ cannot be ordered in dispersion, first choose γ_1 and γ_2 such that $0 < \gamma_1 < \gamma_2$ and then let $X \sim F_{\gamma_1}$ and $Y \sim F_{\gamma_2}$. It suffices to show that neither (i) $\beta^{1/\gamma_1} - \alpha^{1/\gamma_1} \leq \beta^{1/\gamma_2} - \alpha^{1/\gamma_2}$ and (ii) $\beta^{1/\gamma_1} - \alpha^{1/\gamma_1} \geq \beta^{1/\gamma_2} - \alpha^{1/\gamma_2}$ holds for all percentiles α and β with $\alpha < \beta$. But (i) does not hold for $\alpha = 0$ and $\beta \in (0,1)$ and (ii) does not hold for $\alpha \in (0,1)$ and $\beta = 1$.

contract auctions. That is, the addition of unobservable costs is enough to show that the sharing rate does not always induce an ordering in dispersion on the reservation values.

One might think that if c_i and d_i are drawn independently from one another that $v_{i|s} = (1-s)c_i + d_i$ would become more dispersed as s decreases. But even in the extreme case of $s = 0$, where the variance of $c_i + d_i$ clearly exceeds the variance of d_i, it is not generally the case that d_i is smaller in dispersion than $c_i + d_i$. But from Proposition 5, it is the ordering in dispersion—not the ordering of variance—that causes the ordering of information rent. The following proposition shows when adding a random variable to each of two random variables that are ordered in dispersion will preserve that order in dispersion.

Proposition 6 *Let Z be a random variable (with a positive, twice differentiable density function) that is independent from X and Y, and let $X \leq_{disp} Y$. Then $X + Z \leq_{disp} Y + Z$ if and only if the density function of Z is log-concave.*

Proof. See page 86 of Lewis and Thompson (1981). ■

Some distributions with log-concave densities are the normal, Weibull (with shape parameter $r \geq 1$), gamma (with shape parameter $r > 1$), and the uniform. Distributions without log-concave densities include the Pareto, Weibull (with shape parameter $r < 1$), gamma (with shape parameter $r < 1$), reciprocal of gamma, and the Student's t. Applied to our cost sharing model, Proposition 6 leads to the following useful corollary.

Corollary 7 *If d_i is log-concave and independent of c_i, then information rent is decreasing in s.*

Proof. Take $s_1 < s_2$. Lemma 4 implies that $(1-s_2)c_i \leq_{disp} (1-s_1)c_i$ since $1 - s_2 < 1 - s_1$. Proposition 6 then implies that $v_{i|s_2} \leq_{disp} v_{i|s_1}$, so that by Proposition 5, there are less information rents with s_2. ■

Corollary 7 tells us that if c_i and d_i are independent, then the inclusion of unobservable costs can negate the ordering in dispersion induced by McAfee and McMillan's (1986) model, thus allowing for counterexamples to their conventional wisdom. In fact, even more counterexamples are possible if c_i and d_i are correlated, which as Samuelson (1986, 1987) point out, is the far more likely scenario.

An agent with lower observable cost is likely to also have a higher unobservable (opportunity) cost of taking the project if that agent can also do other projects more cheaply than his rivals. Our example from the introduction suffices to show that increasing the sharing rate can actually increase the information rent in the case of correlated observable and unobservable costs. Specifically, suppose that $F(c_i) = c_i^{\sqrt{2}/2}$ on $[0,1]$ and that d_i is distributed such that the convolution of c_i and d_i is distributed by $H_1(v_{i|1}) = v_{i|1}^{(\sqrt{5}-1)/2}$ on $[0,1]$. Then as we have shown in the introductory example, $var(c_i) < var(v_{i|1})$, but $IR_{c_i} > IR_{v_{i|1}}$. That is, information rents are lower with $s = 0$ than with $s = 1$, so that the principal pays lower information rents by letting the agents differ by the full amount of their private costs.[7]

3 Conclusion

Our paper illustrates that dispersion is the driving force between information rents and the distribution of bidder values. Several empirical studies have found that bidder valuations are distributed log-normally.[8] This relates to our findings because log-normal distributions cannot be ordered in dispersion unless they have identical variance parameters (see Lewis and Thompson, 1981). Thus, it is possible that measures aimed at reducing that variance will not necessarily decrease the bidder information rent as previously believed. Our results also apply to the recent empirical auction literature dealing with unobserved heterogeneity.[9] Roberts (2008) and Krasnokutskaya (2004) employ Monte Carlo simulations to analyze the effect of unobserved heterogeneity on the estimation of information rent. Their simulations suggest that failing to account for unobserved heterogeneity causes an upward bias in the estimation. Applying our results, if the distribution of bidders' private values has a log-concave density, then ignoring unobserved heterogeneity will cause the researcher to overestimate information rent.

[7] Similarly, Samuelson (1987) remarks that if each agents' observable and unobservable costs are perfectly negatively correlated, the information rents may be reduced by reducing the sharing rate.

[8] See Baldwin et al. (1997) and Vuong et al. (1995).

[9] See Athey, Levin, and Seira (2004), Bajari and Ye (2003), Campo, Perrigne, and Vuong (2003), Decarolis (2008), and Hong and Shum (2002).

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